ST2-Tutorübung – Lösung zu Blatt 5

1. Fallunterscheidungen in linearen Systemen zweiten Grades

a) $$\lambda_{1,2}=\frac{sp(\boldsymbol{A})}{2}\pm\sqrt{\left(\frac{sp(\boldsymbol{A})}{2}\right)^2-det\,\boldsymbol{A}}=\frac{-6}{2}\pm\sqrt{3^2-(-7+12)}=-3\pm\sqrt{9-5}=-3\pm2=\begin{cases}-1\\-5\end{cases}$$

b) Stabil, weil alle Eigenwerte einen Realteil < 0 haben.

c) Transformation auf Normalform:

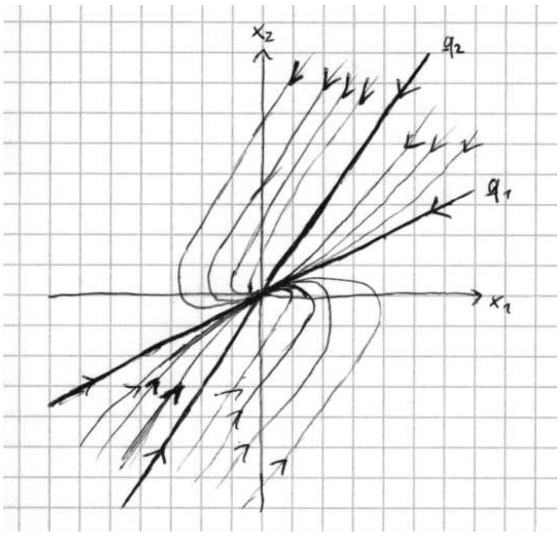
$$\begin{pmatrix}-a_{12}\\a_{11}-\lambda_1\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}\sim\begin{pmatrix}2\\1\end{pmatrix}=\boldsymbol{q_1}\qquad\begin{pmatrix}-a_{12}\\a_{11}-\lambda_2\end{pmatrix}=\begin{pmatrix}4\\6\end{pmatrix}\sim\begin{pmatrix}2\\3\end{pmatrix}=\boldsymbol{q_2}\quad\Rightarrow\ \boldsymbol{Q}=\begin{pmatrix}2&2\\1&3\end{pmatrix}$$

$$\boldsymbol{Q}^{-1}=\frac{1}{6-2}\begin{pmatrix}3&-2\\-1&2\end{pmatrix}=\begin{pmatrix}\frac{3}{4}&-\frac{1}{2}\\-\frac{1}{4}&\frac{1}{2}\end{pmatrix}\qquad\boldsymbol{\Lambda}=diag(\lambda_{1,}\lambda_2)=\begin{pmatrix}-1&0\\0&-5\end{pmatrix}=\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}$$

$$\boldsymbol{\xi}=\boldsymbol{Q}^{-1}\boldsymbol{x}\ \Rightarrow\ \boldsymbol{x}=\boldsymbol{Q}\boldsymbol{\xi}$$

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}\ \Rightarrow\ \boldsymbol{Q}\dot{\xi}=\boldsymbol{A}\boldsymbol{Q}\xi\ \Rightarrow\ \dot{\xi}=\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}\xi\ \Rightarrow\ \dot{\xi}=\boldsymbol{\Lambda}\xi\ \Rightarrow\ \dot{\xi}=\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}\xi\ \Rightarrow\ \dot{\xi}=\begin{pmatrix}-1&0\\0&-5\end{pmatrix}\xi$$

$$\dot{\xi}_1=\lambda_1\xi_1=-\xi_1$$
$$\dot{\xi}_2=\lambda_2\xi_2=-5\xi_2$$



d) $$\boldsymbol{\xi}=e^{\boldsymbol{\Lambda}t}\boldsymbol{\xi_0}=\begin{pmatrix}e^{\lambda_1 t}&0\\0&e^{\lambda_2 t}\end{pmatrix}\begin{pmatrix}\xi_{01}\\\xi_{02}\end{pmatrix}=\begin{pmatrix}e^{\lambda_1 t}\xi_{01}\\e^{\lambda_2 t}\xi_{02}\end{pmatrix}$$

oder direkt aus den entkoppelten Einzelgleichungen:

$$\xi_1=e^{\lambda_1 t}\cdot\xi_{01}$$
$$\xi_2=e^{\lambda_2 t}\cdot\xi_{02}$$

e) $$\boldsymbol{x}=\boldsymbol{Q}\xi=\boldsymbol{q_1}\cdot\xi_1+\boldsymbol{q_2}\cdot\xi_2=e^{-t}\xi_{01}\begin{pmatrix}2\\1\end{pmatrix}+e^{-5t}\xi_{02}\begin{pmatrix}2\\3\end{pmatrix}$$

f) siehe Skizze, „stabiler Knoten“

g) $$\lambda_{1,2}=\frac{sp(\boldsymbol{A})}{2}\pm\sqrt{\left(\frac{sp(\boldsymbol{A})}{2}\right)^2-det\,\boldsymbol{A}}=\frac{-6}{2}\pm\sqrt{3^2-(8+1)}=-3\pm\sqrt{9-9}=-3\pm0=-3=:\lambda$$

Zwei gleiche Eigenwerte $\Rightarrow$ Transformation auf Jordan-Normalform:

$$\boldsymbol{q_1}'=\begin{pmatrix}-a_{12}\\a_{11}-\lambda\end{pmatrix}=\begin{pmatrix}-1\\-2+3\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}\qquad\boldsymbol{q_2}'=\begin{pmatrix}-a_{12}\\a_{11}-\lambda-1\end{pmatrix}=\begin{pmatrix}-1\\-2+3-1\end{pmatrix}\sim\begin{pmatrix}-1\\0\end{pmatrix}$$

$$\Rightarrow\ \boldsymbol{Q}'=\begin{pmatrix}-1&-1\\1&0\end{pmatrix}\qquad\Rightarrow\ \boldsymbol{Q}'^{-1}=\frac{1}{0+1}\begin{pmatrix}0&1\\-1&-1\end{pmatrix}=\begin{pmatrix}0&1\\-1&-1\end{pmatrix}$$

$$\boldsymbol{J}=\boldsymbol{Q}'^{-1}\boldsymbol{A}\boldsymbol{Q}'=\begin{pmatrix}0&1\\-1&-1\end{pmatrix}\begin{pmatrix}-2&1\\-1&-4\end{pmatrix}\begin{pmatrix}-1&-1\\1&0\end{pmatrix}=\begin{pmatrix}-3&1\\0&-3\end{pmatrix}$$

oder direkt: $\boldsymbol{J}=\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}=\begin{pmatrix}-3 & 1\\ 0 & -3\end{pmatrix}$

$\xi=\boldsymbol{Q}'^{-1}\boldsymbol{x} \;\Rightarrow\; \boldsymbol{x}=\boldsymbol{Q}'\xi$

$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x} \;\Rightarrow\; \boldsymbol{Q}'\dot{\xi}=\boldsymbol{A}\boldsymbol{Q}'\xi \;\Rightarrow\; \dot{\xi}=\boldsymbol{Q}'^{-1}\boldsymbol{A}\boldsymbol{Q}'\xi \;\Rightarrow\; \dot{\xi}=\boldsymbol{J}\xi \;\Rightarrow\; \dot{\xi}=\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\xi=\begin{pmatrix}-3 & 1\\ 0 & -3\end{pmatrix}\xi$

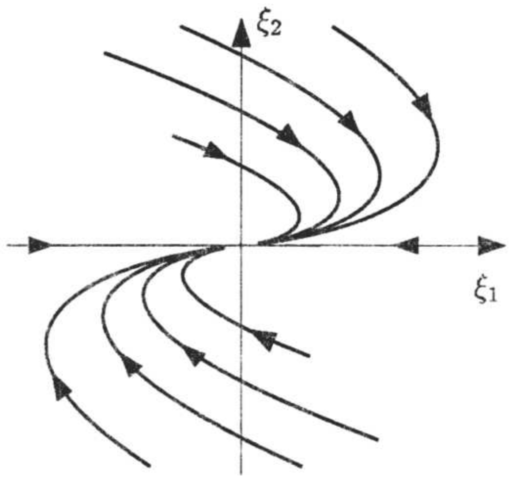
$\dot{\xi}_1=\lambda\xi_1+\xi_2=-3\xi_1+\xi_2$
$\dot{\xi}_2=\lambda\xi_2=-3\xi_2$

h) laut Skript: $\xi=e^{\boldsymbol{J}t}\xi_{\boldsymbol{0}}=\begin{pmatrix}e^{\lambda t} & t\cdot e^{\lambda t}\\ 0 & e^{\lambda t}\end{pmatrix}\begin{pmatrix}\xi_{01}\\ \xi_{02}\end{pmatrix}=\begin{pmatrix}e^{-3t} & t e^{-3t}\\ 0 & e^{-3t}\end{pmatrix}\begin{pmatrix}\xi_{01}\\ \xi_{02}\end{pmatrix}=\begin{pmatrix}e^{-3t}\xi_{01}+t e^{-3t}\xi_{02}\\ e^{-3t}\xi_{02}\end{pmatrix}$

oder aus den Einzelgleichungen:
$\xi_2=e^{\lambda t}\cdot\xi_{02}=e^{-3t}\cdot\xi_{02}$
Betrachte ξ_2 als allgemeine Erregung für System ersten Grades ξ_1 :

$$\xi_1=e^{\lambda t}\cdot\xi_{01}+\int_0^t\xi_2(t')e^{\lambda(t-t')}dt'=e^{\lambda t}\cdot\xi_{01}+\int_0^t e^{\lambda t'}\xi_{02}e^{\lambda t}e^{-\lambda t'}dt'=e^{\lambda t}\cdot\xi_{01}+\int_0^t e^{\lambda t}\xi_{02}dt'$$
$$=e^{\lambda t}\cdot\xi_{01}+e^{\lambda t}\xi_{02}[t']_0^t=e^{\lambda t}\cdot\xi_{01}+t e^{\lambda t}\xi_{02}=e^{-3t}\cdot\xi_{01}+t e^{-3t}\xi_{02}$$

i)



j) aperiodisch gedämpfte Schwingung

k) $\lambda_{1,2}=\frac{sp(\boldsymbol{A})}{2}\pm\sqrt{\left(\frac{sp(\boldsymbol{A})}{2}\right)^2-det\,\boldsymbol{A}}=\frac{4}{2}\pm\sqrt{2^2-(-5+10)}=2\pm\sqrt{4-5}=2\pm\sqrt{-1}=2\pm j$
Komplexe Eigenwerte $\Rightarrow$ Strudel oder Wirbel
Realteil > 0 $\Rightarrow$ instabil
Phasenportrait also: instabiler Strudel

l) Geeignet ist hier die reellwertige Normalform.